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The h-p Version of the Finite Element Method in the Plate Modelling Problem

by

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Technical Note BN-1115

November 1990

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Note BN-1115	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): The h-p Version of the Finite Element Method in the Plate Modelling Problem	5. TYPE OF REPORT & PERIOD COVERED: Final Life of Contract
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): I. Babuska[1] and L. Li	8. CONTRACT OR GRANT NUMBER(s): [1]ONR/N00014-90-J-1030 NSF/CCR-88-20279
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Department of the Navy, Office of Naval Research, Arlington, VA 22217	12. REPORT DATE: November 1990
	13. NUMBER OF PAGES: 16
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Plates, Reissner-Mindlin model, h-p verion of finite element method

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

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The h-p version of the finite element method in the plate modeling problem

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MD90-40-IBLL

TR90-40

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Key words: Plates, Reissner-Mindlin model, h-p version of finite element method

1. Introduction.

The h-p version of the finite element method was developed during the last 10 years. The h-p version programs MSC/PROBE, FIESTA are on the market and the h-p version research code STRIPE (Aeronautical Institute of Sweden) is used in industry as well.

The h-p code provides large flexibility. It allows hierarchical modeling of the plates and shells, avoids the locking problems and provides tools for effective quality control.

This paper addresses a natural hierarchical modeling which the h-p version provides, including the Reissner-Mindlin model. In addition it shows that the h-p version avoids the problem of locking which arises in the standard finite element method.

2. The plate problem and its hierarchical models.

Let $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ (x_1,x_2) \subset \omega, -\frac{d}{2} < x_3 < \frac{d}{2}\}$ be the three dimensional plate ω with the thickness d. Further we let $S = \{x \in \mathbb{R}^3, \ (x_1,x_2) \in \Gamma, -\frac{d}{2} < x_3 < \frac{d}{2}\}$ and $R_{\pm} = \{ x \in \mathbb{R}^3, \ (x_1,x_2) \in \omega, \ x_3 = \pm\frac{d}{2}\}$ where Γ denotes the boundary of ω.

We will consider the three dimensional elasticity problem on Ω with the Hooke's Law

$$(2.1)\qquad \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix}$$

where σ_{ij}, τ_{ij} are the normal and tangential stresses, respectively, ε_{ij}, γ_{ij} are the strain components, and u_i, $i = 1,2,3$ $u = (u_1, u_2, u_3)$ denote the displacements.

For isotropic materials we have $A = \{a_{ij}\}$, $i,j = 1,\ldots 6$ with

$$a_{11} = a_{22} = a_{33} = \lambda + 2\mu = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)},$$

$$a_{12} = a_{13} = a_{23} = a_{21} = a_{31} = a_{32} = \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

$$a_{44} = a_{55} = a_{66} = \mu = \frac{E}{2(1+\nu)} .$$

λ, μ are the Lamé constants, E is the modulus of elasticity and ν is the Poisson ratio. As usual, the strain energy is

$$(2.2)\qquad \varepsilon^A(u) = \frac{1}{2}\int_\Omega (\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \tau_{12}\gamma_{12} + \tau_{13}\gamma_{13} + \tau_{23}\gamma_{23})dx$$

where the stress-strain relations in (2.2) are governed by (2.1) with the matrix $A = \{a_{ij}\}$. A is used as an upper index in the expression (2.2) to emphasize this dependence.

We will assume that the plate is homogeneous, i.e. that a_{ij} are constants. Further we will assume that half of the load $q(x_1, x_2)$ is on R_+ and half on R_-. The total energy is then

$$(2.3)\qquad G^A(u) = \varepsilon^A(u) - Q(u)$$

where

(2.5) $$Q(u) = \int_\omega \frac{q}{2}\left[u_3(x_1,x_2,\frac{d}{2}) + u_3(x_1,x_2,-\frac{d}{2})\right]dx$$

The exact solution $u = (u_1,u_2,u_3)$ of the plate problem is the minimizer of $G^A(u)$ over the subspace $\mathcal{H}(\Omega) \subset (H^1(\Omega))^3$ where $\mathcal{H}(\Omega)$ constraints $(H^1(\Omega))^3$ on S (not on R_+). The boundary conditions of the plate problem are uniquely characterized by $\mathcal{H}(\Omega)$ (respectively the constraints of $(H^1(\Omega))^3$ on S).

Let now $n = (n_1,n_2,n_3)$, $n_i \geq 0$ integer. Then by the solution of the n-model we will understand the minimizer ${}^Bu^{(n)} = \left[{}^Bu_1^{(n)}, {}^Bu_2^{(n)}, {}^Bu_3^{(n)}\right]$ of $G^B(u)$ over the subspace $\mathcal{H}(n) \subset \mathcal{H}(\Omega)$ of all functions of the form

(2.6) $$u_i^{(n)}(x) = \sum_{j=0}^{n_i} u_{i,j}(x_1,x_2)\left[\frac{x_3 2}{d}\right]^j , \qquad i = 1,2,3$$

and the matrix $B = \{b_{ij}\}$ is used as Hooke's law. In general, $B \neq A$.

Because of our symmetry assumptions we made earlier we have

$$u_{1,j} = u_{2,j} = 0 \qquad \text{for } j \text{ even}$$

$$\text{and} \quad u_{3,j} = 0 \qquad \text{for } j \text{ odd.}$$

The solution ${}^Bu^{(n)}$, is the approximation of the exact solution u of the three dimensional problem.

We can use various n for the approximate solution and study the rate of convergence measured in the energy nor'. as $d \longrightarrow 0$. More precisely, we define

(2.7) $$\varepsilon(d) = \left|\frac{\varepsilon^A(u_d) - \varepsilon^B({}^Bu_d^{(n)})}{\varepsilon^A(u_d)}\right|^{1/2}$$

where by the lower index d we express the dependence of the solution on d. Assuming $B = A$, A is isotropic Hooke's law and the solution is smooth, then

$\varepsilon(d) = Cd^{\alpha}$ where α (the rate of convergence) is shown in Table 3.1 for $n_1 = n_2$. We have to distinguish between the two cases $\nu = 0$ and $\nu > 0$.

Table 3.1 The rate of convergence α

a) $\nu > 0$

$n_1 = n_2$	1	1	3	3
n_3	0	2	2	4
α	0	1	2	3

b) $\nu = 0$

$n_1 = n_2$	1	1	3	3
n_3	0	2	2	4
α	1	1	2	3

For more details see [1].

We see that the model (1,1,0) is not admissible for $\nu \neq 0$ because it does not lead to convergent solutions ($\alpha = 0$). Nevertheless, using a modified A, i.e. replacing A by B, we get the value $\alpha = 1$ again. This modification leads to the well known Reissner-Mindlin model which will be discussed in Section 3.

Let us consider a unit square plate of thickness d, $\Omega = \{ x_1, x_2, x_3 \mid |x_i| < 0.5, \ i = 1,2, \ |x_3| < \frac{d}{2} \}$ and assume that the plate is simply supported, with hard support and is uniformly loaded. We assume that the plate is isotropic, homogeneous, with a Poisson's ratio $\nu = 0.3$ and the modulus of elasticity $E = 10^7$. The solution of the n-model with n small is smooth and essentially has no boundary layer (in contrast to the soft simple support).

In Table 3.2 we show the strain energy for the three dimensional solution, the RM model ($\kappa = 0.87$) and the (1,1,2) model (as discussed later).

Table 3.2 The convergence of the RM and (1,1,2) solutions

d	3DIM	RM, $\kappa = 0.87$	ε_{RM}	(1,1,2)	ε_{112}
0.10	0.242115(-6)	0.245521(-6)	11.8%	0.240433(-6)	8.3%
0.025	0.149115(-4)	0.149256(-4)	3.07%	0.149049(-4)	2.10%
0.01	0.232489(-3)	0.232524(-3)	1.2%	0.232472(-3)	0.85%

By ε_{RM} and ε_{112} we denoted the error defined by (2.7) for the RM and n = (1,1,2) model. We clearly see the predicted rate of convergence.

For more detailed analyses of various boundary conditions we refer to [2].

3. The Reissner-Mindlin model for homogeneous isotropic materials.

The well-known Reissner-Mindlin model is described by the following system of differentiable equations for $\vec{\psi} = (\psi_1, \psi_2)$ and ω:

$$(3.1)\qquad \begin{aligned} &\frac{D}{2}\left[(1-\nu)\Delta\psi_1 + (1+\nu)\frac{\partial}{\partial x_1}(\nabla\cdot\vec{\psi})\right] - \kappa\mu d\left(\psi_1 + \frac{\partial \omega}{\partial x_1}\right) = 0 \\ &\frac{D}{2}\left[(1-\nu)\Delta\psi_2 + (1+\nu)\frac{\partial}{\partial x_2}(\nabla\cdot\vec{\psi})\right] - \kappa\mu d\left(\psi_2 + \frac{\partial \omega}{\partial x_2}\right) = 0 \\ &\kappa\mu d(\Delta w + \vec{\nabla}\cdot\psi) + q = 0 \end{aligned}$$

where

$$D = \frac{Ed^3}{12(1 - \nu^2)}$$

$$\mu = \frac{E}{2(1 + \nu)}$$

and $0 < \kappa \le 1$ is a shear factor.

Here ψ denotes the rotation vector and w is the displacement in the vertical direction.

Further we define

$$
\begin{aligned}
M_{11} &= D\left[\frac{\partial\psi_1}{\partial x_1} + \nu\frac{\partial\psi_2}{\partial x_2}\right]\\
M_{22} &= D\left[\nu\frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2}\right]\\
M_{21} = M_{12} &= \frac{1-\nu}{2} D\left[\frac{\partial\psi_1}{\partial x_2} + \frac{\partial\psi_2}{\partial x_1}\right]\\
Q_{31} &= \kappa\mu d\left[\frac{\partial w}{\partial x_1} + \psi_1\right]\\
Q_{32} &= \kappa\mu d\left[\frac{\partial w}{\partial x_2} + \psi_2\right]
\end{aligned}
\tag{3.2}
$$

where M_{ij}, $i,j = 1,2$ are the moments and Q_{3j}, $j = 1,2$ are the shear forces.

Let n and s be the unit normal and right oriented tangential vectors to Γ respectively. Denote further $\psi_n = \vec{\psi}\cdot n$, $\psi_s = \vec{\psi}\cdot s$ and define M_{nn}, M_{ss}, and M_{sn} according to (3.2) replacing x_1 by n and x_2 by s similarly for Q_{3n} and Q_{3s}.

At the boundary, three boundary conditions have to be specified. For example:

i) Clamped plate $\quad w = \psi_1 = \psi_2 = 0$

ii) Soft simple support $\quad w = 0,\ M_{nn} = M_{ns} = 0$

iii) Hard simple support $\quad w = \psi_s = 0,\ M_{nn} = 0$

iv) free $\quad M_{nn} = M_{ns} = Q_{3n} = 0$

and analogously the others.

The Reissner-Mindlin (RM) model was derived using various mechanical principles. It has been shown (see e.g. [3]) that as $d \longrightarrow 0$ the solution of the RM model converges to the solution of the three-dimensional model in the (scaled) energy norm.

Let us define $B = \{b_{ij}\}$, $i,j = 1,\dots 6$ with

$$B = \begin{bmatrix} \lambda_1+2\mu_1 & \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_1 & \lambda_1+2\mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1+2\mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa\mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa\mu_1 \end{bmatrix}$$

where

$$\lambda_1 = \frac{E_1\nu_1}{(1+\nu_1)(1-2\nu_1)} \qquad \mu_1 = \frac{E_1}{2(1+\nu_1)}$$

$$\nu_1 = \frac{\nu}{1+\nu} \qquad E_1 = \frac{E(1+2\nu)}{(1+\nu)^2}$$

and denote by ${}^Bu$ the minimizer of $G^B(u)$ (see (2.3)) when u has the form (2.6) with $n_1 = n_2 = 1$, $n_3 = 0$ and $\mathcal{H}(\Omega)$ is defined accordingly.

For example in the case of:

i) clamped plate : $\mathcal{H}(\Omega) = \{u \in (H^1(\Omega))^3,\ \ u = 0 \text{ on } S\}$

ii) soft simple support : $\mathcal{H}(\Omega) = \{u \in (H^1(\Omega))^3,\ \ u_3 = 0 \text{ on } S\}$

iii) hard simple support : $\mathcal{H}(\Omega) = \{u \in (H^1(\Omega))^3,\ \ (u_1,u_2)\cdot s = 0,\ u_3 = 0, \text{ on } S\}$

iv) free plate : $\mathcal{H}(\Omega) = \{u \in (H^1(\Omega))^3\}$

Let now ${}^B\sigma_{i,j}$ and ${}^B\tau_{i,j}$ be the stresses computed from ${}^Bu$ by using Hooke's matrix B and define

$${}^BM_{11} = \int_{-d/2}^{d/2} \sigma_{11}x_3dx_3$$

$${}^BM_{22} = \int_{-d/2}^{d/2} \sigma_{22}x_3dx_3$$

(3.3)
$$^{B}M_{12} = \int_{-d/2}^{d/2} \tau_{12}x_3 dx_3$$

$$^{B}Q_{31} = \int_{-d/2}^{d/2} \tau_{31} dx_3$$

$$^{B}Q_{32} = \int_{-d/2}^{d/2} \tau_{32} dx_3$$

Now it is not hard to prove the following:

Theorem 1. We have with (3.2) and (3.3)

$$M_{ij} = {}^{B}M_{ij} \qquad i = 1,2$$

$$Q_{3i} = {}^{B}Q_{3i} \qquad i = 1,2$$

and

$$w = {}^{B}u_{3,0}, \qquad \psi_1 = {}^{B}u_{1,1}, \qquad \psi_2 = {}^{B}u_{2,1}$$

where M_{ij}, $Q_{i,j}$ are the Reissner-Mindlin moments and shear forces from (3.2). The proof is based on the comparison of the bilinear forms for the variational formulation of the RM model and the (1,1,0) model. □

We remark that for $\nu = 0$, the matrices A and B are identical.

4. The Reissner-Mindlin model for homogeneous orthotropic plate.

Let the Hooke's law for the orthotropic material is

$$(4.1) \quad \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{31}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{32}^{-1} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix}$$

where $\nu_{12} = \nu_{21}\frac{E_{11}}{E_{22}}$, $\nu_{13} = \nu_{31}\frac{E_{11}}{E_{33}}$, $\nu_{23} = \nu_{32} = \frac{E_{22}}{E_{33}}$. We can write also (4.1) in the form of (2.1).

Following [4] the system of differential equations analogous to equation (3.1) for $\vec{\psi} = (\psi_1, \psi_2)$ and ω is

$$\frac{d^3}{12}\frac{\partial}{\partial x_1}\left[C_{11}\frac{\partial \psi_1}{\partial x_1} + C_{12}\frac{\partial \psi_2}{\partial x_2}\right] + \frac{d^3}{12}G_{12}\frac{\partial}{\partial x_2}\left[\frac{\partial \psi_2}{\partial x_1} + \frac{\partial \psi_1}{\partial x_2}\right] - \kappa d G_{31}\left[\psi_1 + \frac{\partial \omega}{\partial x_1}\right] = 0$$

$$\frac{d^3}{12}G_{12}\frac{\partial}{\partial x_1}\left[\frac{\partial \psi_2}{\partial x_1} + \frac{\partial \psi_1}{\partial x_2}\right] + \frac{d^3}{12}\frac{\partial}{\partial x_2}\left[C_{21}\frac{\partial \psi_1}{\partial x_1} + C_{22}\frac{\partial \psi_2}{\partial x_2}\right] - \kappa d G_{32}\left[\psi_2 + \frac{\partial \omega}{\partial x_2}\right] = 0$$

$$\kappa d G_{31}\frac{\partial}{\partial x_1}\left[\psi_1 + \frac{\partial \omega}{\partial x_1}\right] + \kappa d G_{32}\frac{\partial}{\partial x_2}\left[\psi_2 + \frac{\partial \omega}{\partial x_2}\right] + q = 0$$

where

$$C = \{C_{ij}\} \quad i, j, = 1,2$$

$$C^{-1} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} \end{bmatrix}$$

The analog to (3.2) is

$$M_{11} = \frac{d^3}{12}\left[C_{11}\frac{\partial\psi_1}{\partial x_1} + C_{12}\frac{\partial\psi_2}{\partial x_2}\right]$$

$$M_{22} = \frac{d^3}{12}\left[C_{21}\frac{\partial\psi_1}{\partial x_1} + C_{22}\frac{\partial\psi_2}{\partial x_2}\right]$$

(4.2)
$$M_{21} = M_{12} = \frac{d^3}{12} G_{12}\left[\frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_1}{\partial x_2}\right]$$

$$Q_{31} = \varkappa d G_{31}\left[\frac{\partial w}{\partial x_1} + \psi_1\right]$$

$$Q_{32} = \varkappa d G_{32}\left[\frac{\partial w}{\partial x_2} + \psi_2\right]$$

and M_{nn}, M_{ns}, M_{ss}, Q_{3n}, Q_{3s}, are defined accordingly

$$B = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa G_{32} \end{bmatrix}$$

Denoting by ψ_1, ψ_2, w the solution of the Reissner-Mindlin equation for given boundary conditions and by ${}^{B}u$ the minimizer of G^B, then, as in the previous section the Theorem 1 also holds. The proof is analogous.

5. <u>The general anisotropic case</u>.

Consider the general case of the Hooke's law defined by (2.1). Then if the matrix $B = \{b_{ij}\}$ is such that

$$b_{ij} = a_{ij} - \frac{a_{i3}a_{3j}}{a_{33}}, \qquad i,j = 1,2,4$$

$$b_{31} = b_{32} = b_{13} = b_{23} = b_{34} = b_{43} = 0$$

$$b_{33} = a_{33}$$

$$b_{ij} = \kappa a_{ij}, \qquad i,j = 5,6$$

Then once more Theorem 1 holds. The proof is analogous.

6. The h-p version of the finite element method.

Let the domain ω be partitioned into two-dimensional straight or curvilinear triangular or quadrilateral (two-dimensional) elements e_i in x_1x_2 plane. Then the three dimensional elements are $e_i \times (-\frac{d}{2}, \frac{d}{2})$. The h-p version which is implemented in MSC/PROBE and STRIPE program uses polynomials of degree q (for u_i, $i = 1,2,3$) in the x_3 direction and polynomials of degree p in x_1x_2 plane with $p \geq q$. For quadrilateral elements both programs use the serendipity elements in the x_1x_2 plane.

The elements of the p,q type are used. Then for fixed q and for $p \longrightarrow \infty$ (or when the size of elements e_i converges to zero) we get the solution of the n-model with $n = (q,q,q)$. This model in the case of symmetrical load is equivalent with the model $n = (n_1,n_2,n_3)$, $n_1 = n_2 = 2\left[\frac{q+1}{2}\right] - 1$ and $n_3 = 2\left[\frac{q}{2}\right]$ where by $[a]$ we denote the integral part of a. It is also possible to use different q in different elements. Various material properties, isotropic, anisotropic, are available in the programs. Hence as has been shown in previous sections the RM model is obtained by properly adjusting the Hooke's law matrix.

Let us consider the case of the unit square with hard simple support which is uniformly loaded. We will use the RM and (1,1,2) model. Let us be interested in the locking problem i.e. the convergence of the h-version using x_1x_2 plane elements of various degrees.

Consider a sequence of uniform square meshes with element size h. In Figure 6.1abc we show the error (measured in the energy norm) as a function of h for various h and various degrees of

elements p. In the figure we also depict the slope of the asymptotic error rate h^{α}.

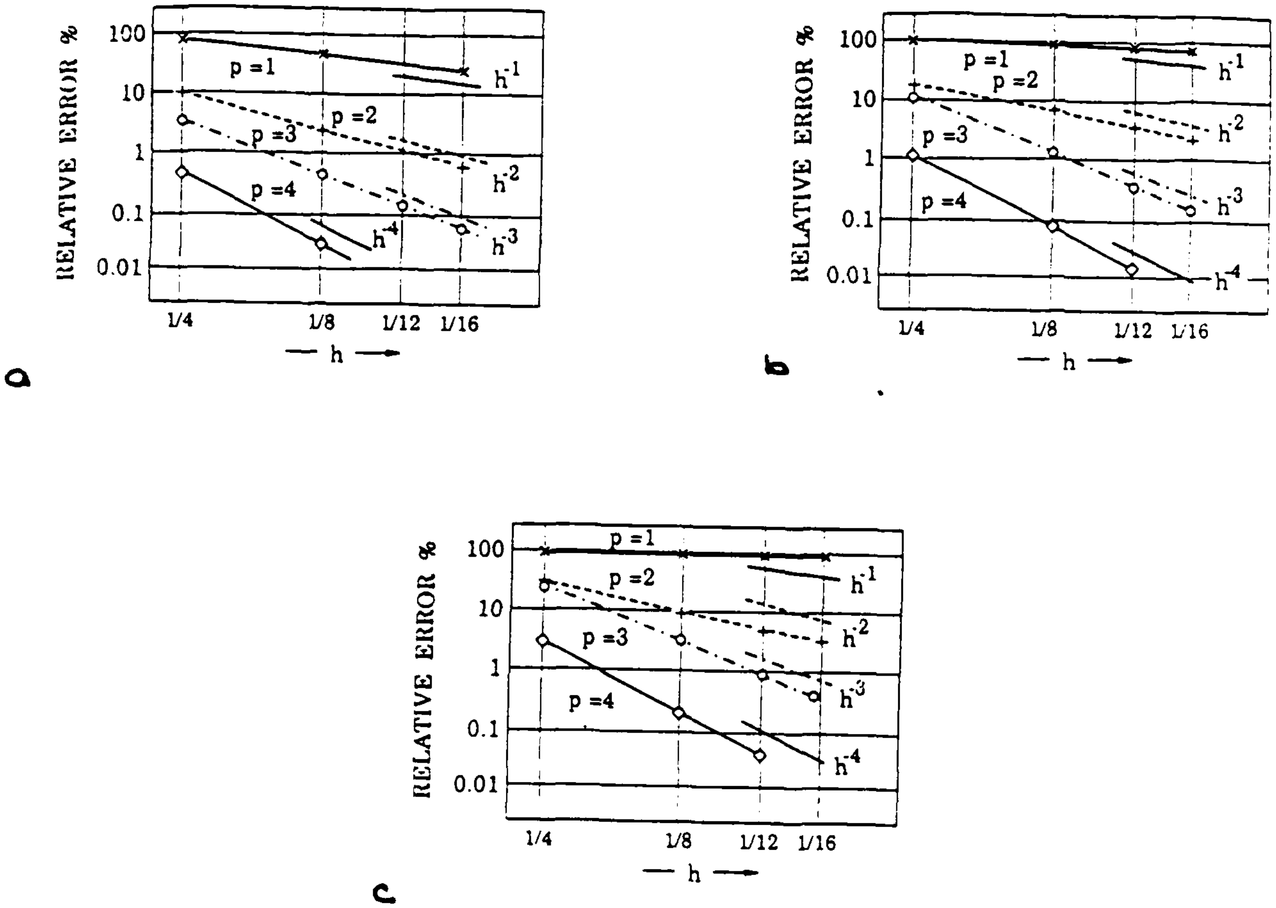

Figure 6.1 The convergence of the h-version for the RM model. a: d = 0.1, b: d = 0.025, c: d = 0.01

We see that for $d = 0.1$ the rate of convergence is $\alpha(p) = p$. For $d = 0.01$ we see $\alpha(1) \approx 0$, $\alpha(p) < p$, $p = 2,3$, but $\alpha(4) = 4$ as for $d = 0.01$. The lower rate effect is the typical locking phenomenon. For $d = 0.025$ we see locking for $p = 1,2$ but for $p = 3$ the locking is essentially not present.

Figure 6.2ab shows the typical locking for the p-version. We see here once more that for increasing p there is no difference between d = 0.01 and d = 0.1 i.e. that the p-version is locking free.

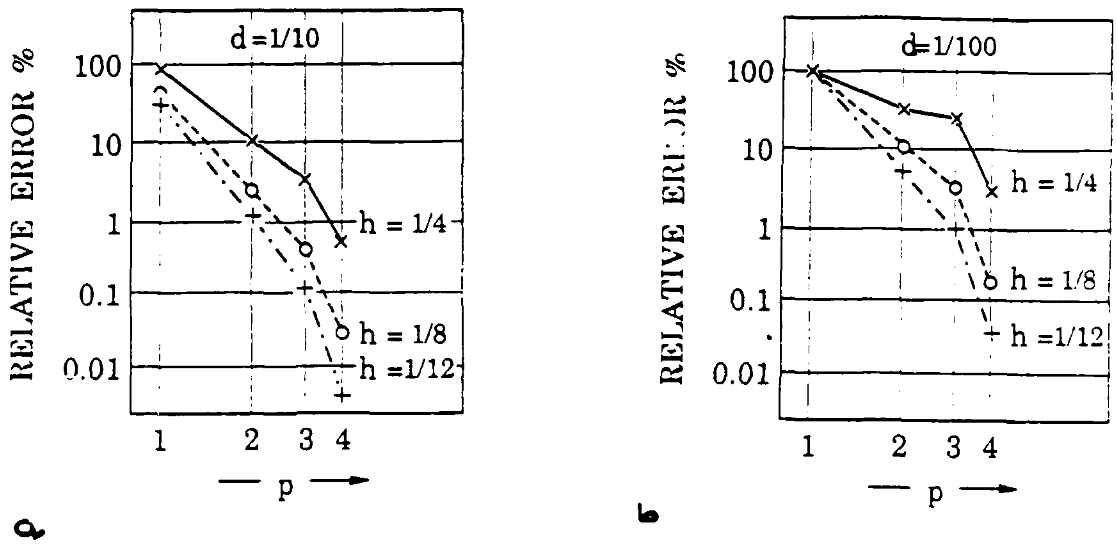

Figure 6.2 The convergence of the p-version for the RM model. a: d = 0.1, b: d = 0.01

Figure 6.3abc shows performance of the h version for the model (1,1,2) as a function of p, analogously as before. We see quite analogous results. Hence for $p \geq 4$ the method is locking free also here.

In [5] we have addressed the general model of locking. We have underlined that often the deterioration of the finite element method is the combined effect of locking and decreasing regularity of the solution. Hence it is essential to guarantee that the regularity of the solutions is essentially uniform with respect to the parameter (in our case d) of interest. We have for this reason selected the hard simple support.

In the case of the model (3,3,4) the solution of the hard simple support is not any more sufficiently smooth any more and the locking is not the main factor influencing the convergence rate. This leads to the exactly

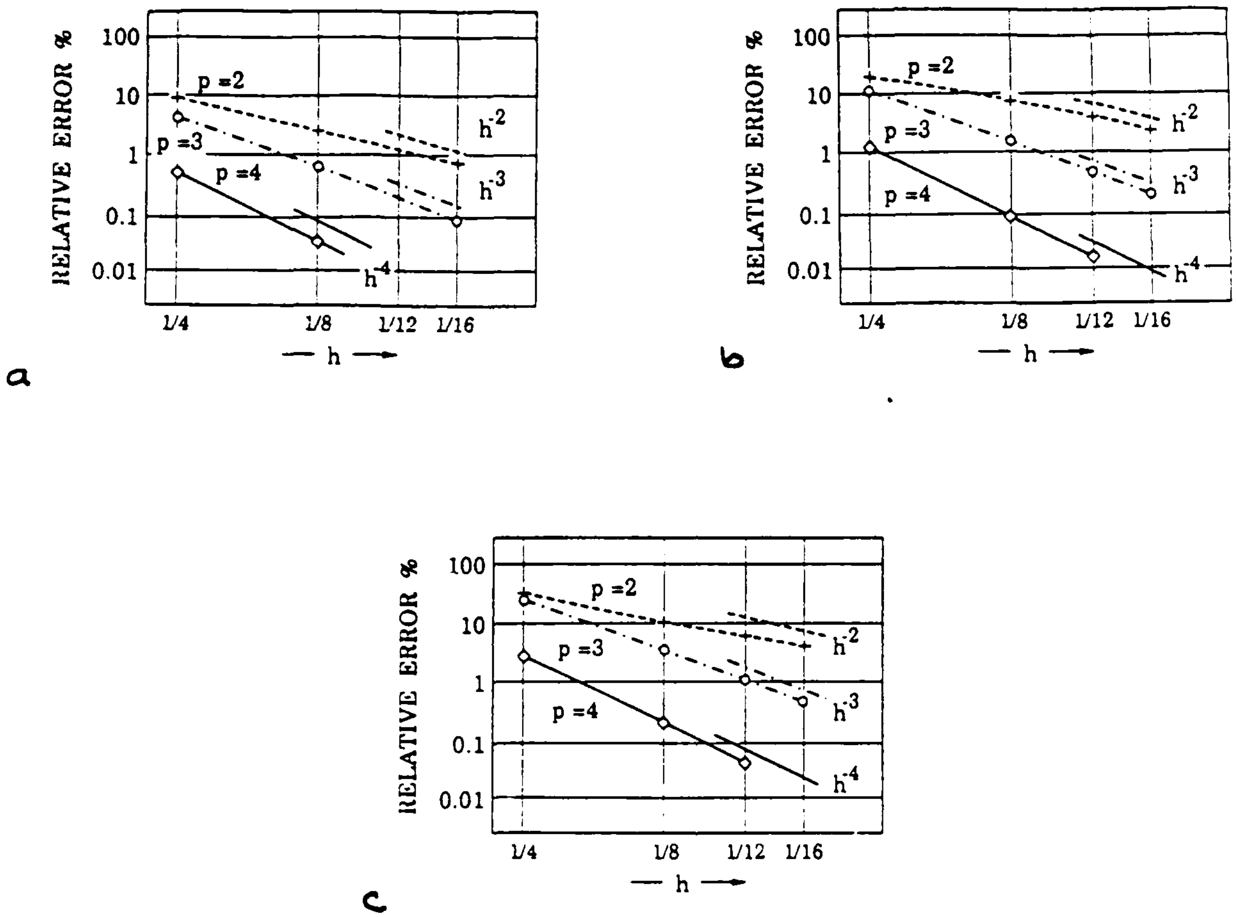

Figure 6.3 The convergence of the h-version for the (1,1,2) model. a: d = 0.1, b: d = 0.025 d = 0.01.

opposite character we have seen above. For d larger we see smaller rate of convergence than for d smaller. In Figure 6.4 we show the error as a function of h for p = 4. It clearly indicates the effect of the regularity of the solutions for various d. The effect of the unsmoothness of the solution for d = 0.1 can be seen by comparing the accuracy obtained for different meshes. In the Table 6.1 we show the error for various meshes of 16 elements for $d = 0.1$ and $d = 0.01$. These meshes are characterized by coordinates $x_1 \geq 0$.

We see that for $d = 0.1$ the uniform mesh (No. 1) leads to a larger error while a refined mesh (No. 2) leads to a smaller one. On the other hand the uniform mesh (with 16 elements) for $d = 0.01$ is the optimal one. This clearly explains Figure 6.4 where the rate of convergence for $d = 0.1$ is smaller (in our range) than for $d = 0.01$, which is opposite what has been seen for the RM and (1,1,2) model before.

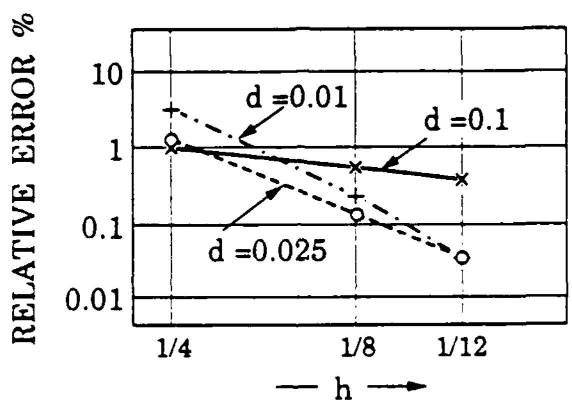

Figure 6.4 The convergence of the h-version for the (3,3,4) model.

Table 6.1 The energy error for the model (3,3,4) and element of degree 4

	COORDINATES					ERROR %	
No	x_0	x_1	x_2	x_3	x_4	d = 0.1	d = 0.01
1	0	0.125	0.25	0.375	0.50	0.482	0.209
2	0	0.20	0.38	0.45	0.50	0.273	0.434
3	0	0.16	0.32	0.43	0.50	0.315	---
4	0	0.30	0.45	0.49	0.50	0.407	---

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